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TENTAMEN RELATIVISTIC QUANTUM MECHANICS

Monday 02-02-2009, 09.00-12.00

On the first sheet write your name, address and student number. Write your name on all other sheets.

This examination consists of four problems, with in total 17 parts. The 17 parts carry equal weight in determining the final result of this examination.

$\hbar = c = 1$. The standard representation of the 4×4 Dirac gamma-matrices is given by:

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \qquad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \qquad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}.$$

PROBLEM 1

The Lagrangian density for the Dirac field is

$$\mathcal{L} = \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x).$$

1(a) Define the canonical momentum corresponding to the field ψ, and show that it equals

$$\pi(t, \vec{x}) = i\psi^\dagger(t, \vec{x}).$$

1(b) The Hamiltonian is defined as

$$H = \int d^3x\, (\pi(t, \vec{x})\partial_0\psi(t, \vec{x}) - \mathcal{L}).$$

Show that this equals

$$H = -\int d^3x\, \bar{\psi}(i\gamma^k\partial_k - m)\psi.$$

if $\psi(x)$ satisfies the Dirac equation.

1(c) The invariance of $\mathcal{L}$ under transformations

$$\psi \to \psi' = e^{i\theta}\psi$$

gives rise to a current $j^\mu \equiv \bar{\psi}\gamma^\mu\psi$. Show that, if the Dirac equation for ψ (and $\bar{\psi}$) holds, j^μ satisfies

$$\partial_\mu j^\mu = 0.$$

1(d) Show that

$$Q = \int d^3x\, j^0$$

is constant in time if ψ and its spatial derivatives go sufficiently fast to zero at large $|\vec{x}|$.

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PROBLEM 2

Consider the following Lagrangian density for a quantum field theory involving two scalar fields ϕ_1 and ϕ_2:

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + a\partial_\mu\phi_1\partial^\mu\phi_2 + \tfrac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - 2m^2\phi_1\phi_2 .$$

where a is a constant.

2(a) Determine the equations of motion for ϕ_1 and ϕ_2.

2(b) What are the canonical momenta π_1 and π_2 associated to ϕ_1 and ϕ_2?

2(c) Express the Hamiltonian in terms of the canonical coordinates and momenta.

2(d) Give the result of 2(c) for the special case $a = 0$.

2(e) Show that for $a = 1$ the equations of motion imply that $\phi_1 = \phi_2$. What is in this case the mass of the field $\phi \equiv \phi_1 + \phi_2$?

PROBLEM 3

The Hamiltonian corresponding to the Dirac equation in an external electromagnetic field is

$$H = \gamma^0\vec{\gamma}\cdot\vec{\pi} + eA_0(x) + m\gamma^0 , \tag{3.1}$$

where $\pi_k = p_k + eA_k(x) = -i\big(\partial_k + ieA_k(x)\big)$. We define the **helicity** as the component of the spin of the electron in the direction of the momentum

$$\Sigma = S^k\pi_k , \tag{3.2}$$

where the spin $\vec{S} = \frac{1}{2}\gamma^5\gamma^0\vec{\gamma}$.

In the first two parts of this problem we set $e = 0$, i.e., we switch off the electromagnetic field.

3(a) Show that $[\gamma^0, \Sigma] = 0$.

3(b) Show that the helicity is conserved: $[H, \Sigma] = 0$.

Now we set $e \neq 0$ and switch on the electromagnetic field, with vector potential

$$A_0(x) = 0 , \quad A_k(x) = \tfrac{1}{2}\epsilon_{klm}x^l B^m , \tag{3.3}$$

where $\vec{B}$ is a constant vector.

3(c) Show that the electric field corresponding to this vector potential vanishes, and that the magnetic field is given by $\vec{B}$.

3(d) Show that the helicity (which now also contains A_k, see eq. (3.2)!) is still conserved.

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PROBLEM 4

In a scattering process the particles in the initial and final state are considered to be free relativistic particles.

Consider the elastic scattering process between two electrons:

$$e_1 + e_2 \to e_3 + e_4,$$

with four-momenta k_1^μ, k_2^μ, k_3^μ, k_4^μ. The spatial momenta satisfy (center of mass frame)

$$\vec{k}_1 + \vec{k}_2 = 0\,.$$

The electrons have mass m.

4(a) What is the value of $(k_i)^\mu (k_i)_\mu$ for $i = 1, \ldots, 4$?

4(b) What is the value of $\vec{k}_3 + \vec{k}_4$?

4(c) Show that the energies of the four electrons are equal.

4(d) Choose a coordinate system such that the spatial momentum of e_1 is

$$\vec{k}_1 = (k, 0, 0)\,.$$

We now perform a Lorentz boost in the x^1 direction. On the momenta this acts as on the coordinates:

$$k_i^{0\,\prime} = \gamma(k_i^0 - v k_i^1)\,, \quad k_i^{1\,\prime} = \gamma(-v k_i^0 + k_i^1)\,, \quad k_i^{2\,\prime} = k_i^2\,, \quad k_i^{3\,\prime} = k_i^3\,,$$

where the index $i = 1, \ldots, 4$ indicates the four electrons and $\gamma = 1/\sqrt{1 - v^2}$. Calculate $\vec{k}_3{}' + \vec{k}_4{}'$ in this new coordinate system.